Muller Example:

The IVP:

$$x' = f(t,x), \quad x(0) = 0,$$

where

$$f(t,x)=\begin{cases} 0, & t \le 0,\ -\infty < x < \infty \\ 2t, & t > 0,\ x < 0 \\ 2t - \dfrac{4}{t}x, & t > 0,\ 0 \le x < t^2 \\ -2t, & t > 0,\ t^2 \le x < \infty \end{cases}.$$

Solution: a) To show $\lim\limits_{\substack{t\to 0\\ x\to 0}} f(t,x) = 0 = f(0,0)$, we first note that

$$|2t - \frac{4}{t}x| \le 2|t| + 4\frac{x}{|t|} \le 2|t| + 4\frac{t^2}{|t|} = 2|t| + 4|t| = 6|t| \to 0 \quad \text{as} \quad t \to 0.$$

Then, we know that $\lim\limits_{\substack{t\to 0\\ x\to 0}} f(t,x) = 0 = f(0,0)$ by the definition of $f(t,x)$. Therefore, the existence of solution is assured by the Peano theorem.

b) Since

$$\frac{\partial f}{\partial x}(t,x) = \begin{cases} 0, & t \le 0,\ -\infty < x < \infty \\ 0, & t > 0,\ x < 0 \\ -\dfrac{4}{t}, & t > 0,\ 0 \le x < t^2 \\ 0, & t > 0,\ t^2 \le x < \infty \end{cases}$$

exists on any neighborhood of the origin except for the origin. However,

$$\lim_{\substack{t\to 0\\ x\to 0}} \frac{\partial f}{\partial x}(t,x) = \infty.$$

This shows that $f(t,x)$ doesn't satisfy a Lipschitz condition in any neighborhood of the origin. (Why? The details leave for students)

c) Applying Picard iteration with $x_0(t) \equiv 0$, we have

$$x_{2m-1}(t) = t^2, \quad x_{2m}(t) = -t^2, \quad m \in N^+,$$

which shows that the Picard sequence $\{x_n(t)\}$ is not convergent uniformly on $t \in [-h,h]$. Although its two subsequences do converge uniformly on $t \in [-h,h]$, they are not convergent uniformly to the solution because

$$x'_{2m-1}(t) = 2t \neq f(t, t^2), \quad x'_{2m}(t) = -2t \neq f(t, -t^2).$$

d) If there exist two solutions $x_1(t)$ and $x_2(t)$, defined on $t \in [0, h]$, where $0 < h < \infty$, then $\delta(t) = (x_1(t) - x_2(t))^2$ satisfies

$$\delta(0) = 0, \quad \delta(t) \geq 0, \quad t \in [0, h].$$

Then

$$\delta'(t) = 2[x_1'(t) - x_1'(t)][x_1(t) - x_1(t)] = 2[f(t, x_1(t)) - f(t, x_2(t))][x_1(t) - x_1(t)].$$

Since $f(t, x)$ is not increased on x by the definition of $f(t, x)$, we have

$$[f(t, x_1(t)) - f(t, x_2(t))][x_1(t) - x_1(t)] \leq 0, \quad t \in [0, h].$$

It yields $\delta'(t) \leq 0$, so $\delta(t) \leq 0$, $t \in [0, h]$. So it must be $\delta(t) \equiv 0$ for $t \in [0, h]$. That is, $x_1(t) \equiv x_2(t)$ for $t \in [0, h]$. It is similar to show for $t \in [-h, 0]$. The uniqueness is done.

1) In the domain of $t \leq 0$, $-\infty < x < \infty$ and $t > 0$, $x < 0$, the solution is given by

$$x(t) = \begin{cases} 0, & t \leq 0 \\ -t^2, & t > 0 \end{cases}.$$

2) In the domain of $t \leq 0$, $-\infty < x < \infty$ and $t > 0$, $0 \leq x < t^2$, the solution is given by

$$x(t) = \begin{cases} 0, & t \leq 0 \\ \frac{2}{3}t^2, & t > 0 \end{cases}.$$

3) In the domain of $t \leq 0$, $-\infty < x < \infty$ and $t > 0$, $t^2 \leq x$, the solution is given by

$$x(t) = \begin{cases} 0, & t \leq 0 \\ t^2, & t > 0 \end{cases}. \qquad \square$$

Remark: This example is famous, imitated by M. Muller in 1923. It shows that the conditions that assure the existence and uniqueness of solution is not enough to assure the Picard iteration uniformly convergent or, to the solution although the continuous and Lipschitz conditions can do since we have many other uniqueness condition except for Lipschitz condition.